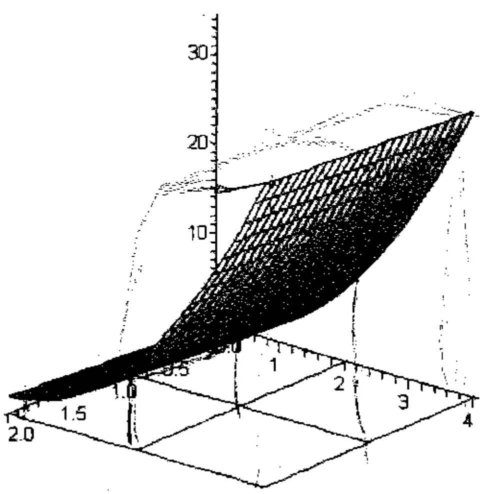
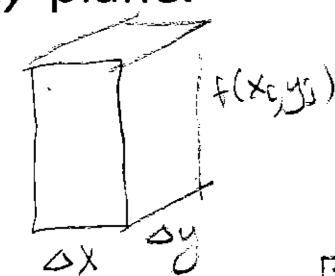
15.1-15.2 Intro to double Integrals

Goal: Give a definition for the volume between a *given surface* and a *given region* on the *xy*-plane.

In all of ch. 15, you are given two things:

1. A surface: z = f(x,y)
2. A region drawn on the *xy*-plane.





Example:

The volume under

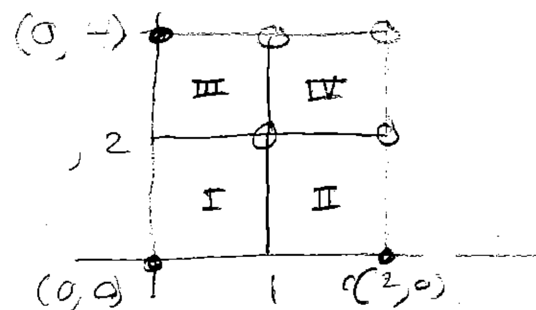
$$z = f(x,y) = x + 2y^2$$

and above

R = [0,2] x [0,4] = {(x,y) : 0 ≤ x ≤ 2, 0 ≤ y ≤ 4 }

(a) Break the region R into m = 2 columns and n = 2 rows; 4 sub-regions;

(a) Approx. using a rectangular box over each region (use *upper-right* endpts).



$\Delta x = \frac{2-0}{2} = 1, \quad \Delta y = \frac{4-0}{2} = 2$

$\Delta A = \Delta x \Delta y = (1)(2) =$ AREA OF "BASE"

$\boxed{\text{I}}$ $f(1,2) = (1) + 2(2)^2 = 9 \Rightarrow \text{VOL} = f(1,2)\Delta A = 9 \cdot 2 = 18$

$\boxed{\text{II}}$ $f(2,2) = (2) + 2(2)^2 = 10 \Rightarrow \text{VOL} = f(2,2)\Delta A = 10 \cdot 2 = 20$

$\boxed{\text{III}}$ $f(1,4) = (1) + 2(4)^2 = 33 \Rightarrow \text{VOL} = f(1,4)\Delta A = 33 \cdot 2 = 66$

$\boxed{\text{IV}}$ $f(2,4) = (2) + 2(4)^2 = 34 \Rightarrow \text{VOL} = f(2,4)\Delta A = 34 \cdot 2 = 68$

TOTAL VOL $\approx 18 + 20 + 66 + 68 = 172$ ← BIG OVER APPROXIMATION

Formally, we define:

$$\iint_R f(x,y)dA = \lim_{m,n\to\infty} \sum_{i=1}^{m}\sum_{j=1}^{n} f(x_{ij}, y_{ij})\Delta A$$

= the `signed' volume between *f(x,y)* and the *xy*-plane over *R*.

If *f(x,y)* is above the *xy*-plane it is positive.
If *f(x,y)* is below the *xy*-plane it is negative.

General Notes and Observations:

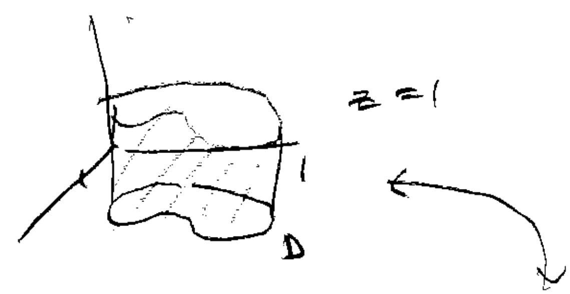
z = *f(x,y)* = height on surface
R = the region on the *xy*-plane

ΔA = area of base = $\Delta x \Delta y = \Delta y \Delta x$
$f(x_{ij}, y_{ij})\Delta A$ = (height)(area of base)
= volume of one approximating box

Units of $\iint_R f(x,y)dA$ are
(units of f(x,y))(units of x)(units of y)

Quick application note:

$$\iint_R 1dA = \text{"Area of R"}\text{, and}$$

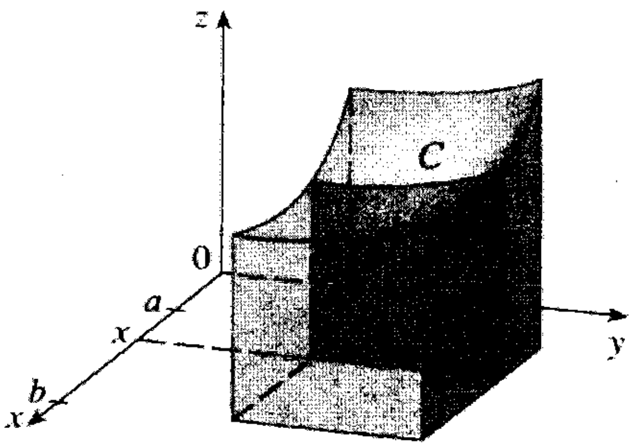


VOLUME = 1 · AREA OF BASE

Iterated Integrals

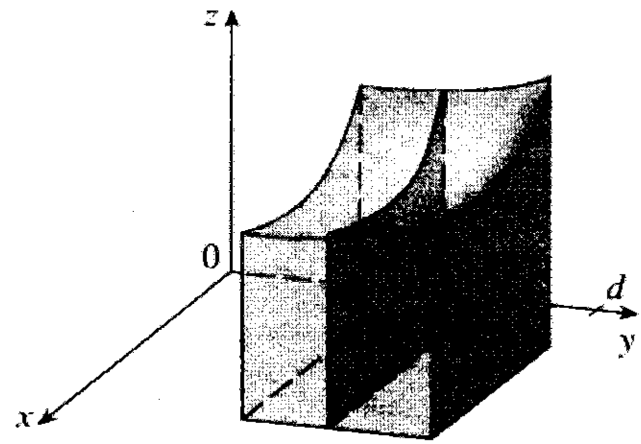
If you fix x: The area under this curve is

$$\int_c^d f(x,y)dy = \text{"cross sectional area under the surface at this fixed } x \text{ value"}$$



If you fix y: The area under this curve

$$\int_a^b f(x,y)dx = \text{"cross sectional area under the surface at this fixed } y \text{ value"}$$



From Math 125,

$$\text{Vol} = \int_a^b \text{Area(x)}dx = \int_a^b \left(\int_c^d f(x,y)dy \right) dx$$

$$\text{Vol} = \int_c^d \text{Area}(y)dy = \int_c^d \left(\int_a^b f(x,y)dx \right) dy$$

Quick Example: Evaluate

$$(a)\ \int_2^6 \left(\int_1^8 y\,dx \right) dy$$

$x=1$

$$= \int_2^6 \left(y \times x \Big|_{x=1}^{x=8} \right) dy$$

CROSS-SECTIONAL AREA WHEN y IS FIXED

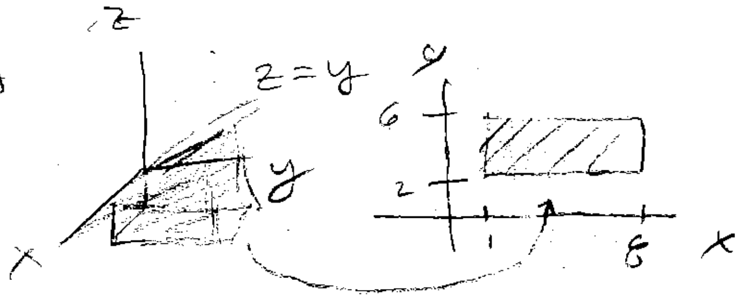
$$= \int_2^6 8y - y\,dy = \int_2^6 7y\,dy$$

$$= \frac{7}{2}y^2\Big|_2^6 = \frac{7}{2}\left((6)^2-(2)^2\right)$$

$$= \frac{7}{2}(36-4) = \frac{7}{2}\cdot 32 = 7\cdot 16$$

$$= \boxed{112}$$

Z=y IS A PLANE



$$(b)\ \int_2^6 \left(\int_1^8 1\,dx \right) dy$$

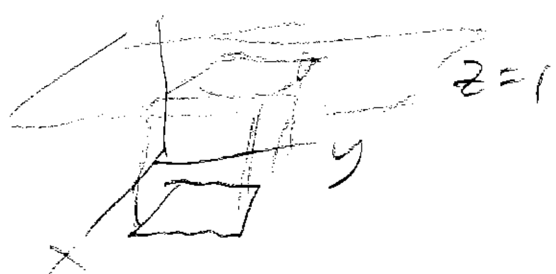
$$= \int_2^6 \left(x \Big|_1^8 \right) dy$$

CROSS-SECTIONAL AREA WHEN y IS FIXED

$$= \int_2^6 8-1\,dy = \int_2^6 7\,dy$$

$$= 7y\Big|_2^6 = 7(6-2)$$

$$= \boxed{28} = \text{AREA OF REGION}$$



Examples (like 15.2 HW):

1. Find the volume under $z = x + 2y^2$ and above $0 \le x \le 2, \quad 0 \le y \le 4$

$$\int_0^4 \left(\int_0^2 x + 2y^2 \, dx \right) dy$$

$$\int_0^4 \left(\frac{1}{2}x^2 + 2y^2 x \Big|_0^2 \right) dy$$

$$\int_0^4 2 + 4y^2 \, dy$$

$$2y + \frac{4}{3}y^3 \Big|_0^4$$

$$8 + \frac{4}{3} \cdot 64 = 8 + \frac{256}{3}$$

$$= 93.\overline{3}$$

OR

$$\int_0^2 \left(\int_0^4 x + 2y^2 \, dy \right) dx$$

$$\int_0^2 \left(xy + \frac{2}{3}y^3 \Big|_0^4 \right) dx$$

$$\int_0^2 4x + \frac{128}{3} \, dx$$

$$2x^2 + \frac{128}{3}x \Big|_0^2$$

$$8 + \frac{256}{3}$$

$$= 93.\overline{3}$$

COMPARE TO OUR APPROXIMATION EARLIER

2. $\int_0^3 \left(\int_0^1 2xy\sqrt{x^2+y^2}\,dx\right)dy$

$2x\underbrace{\sqrt{x^2+1}}$

SUBSTITUTION!!!

$\int_0^3 \left(\int_{y^2}^{1+y^2} \cancel{2x}y \sqrt{u} \frac{1}{\cancel{2x}} du \right) dy$

$u = x^2+y^2$

$du = 2x\,dx$

$\frac{1}{2x} du = dx$

$\int_0^3 \left(y \frac{2}{3} u^{3/2} \Big|_{y^2}^{1+y^2} \right) dy$

$\int_0^3 \frac{2}{3} y (1+y^2)^{3/2} - \frac{2}{3} y \overbrace{(y^2)^{3/2}}^{y^3} dy$

$\int_0^3 \frac{2}{3} y (1+y^2)^{3/2} - \frac{2}{3} y^4 dy$

$-\frac{2}{15} y^5 \Big|_0^3$

$-\frac{2}{15}(3)^5 = -\frac{162}{5}$

$\int_0^3 \frac{2}{3} y (1+y^2)^{3/2} dy$

$u = 1+y^2$

$du = 2y\,dy$

$\frac{1}{2y} du = dy$

$\int_1^{10} \frac{2}{3}\cancel{y}\, u^{3/2} \frac{1}{2\cancel{y}} du$

$\frac{1}{3} \frac{2}{5} u^{5/2} \Big|_1^{10} = \frac{2}{15}(10^{5/2}-1)$

$$\boxed{\frac{2}{15}(10^{5/2}-1) - \frac{162}{5} \approx 9.63}$$

3. Find the double integral of

$$f(x,y) = y\cos(x+y)$$

over the rectangular region

$$0 \le x \le \pi,\ \ 0 \le y \le \pi/2$$

$$\int_0^{\pi/2}\left[\int_0^{\pi} y\cos(x+y)\,dx\right]dy \qquad \underline{\text{Or}} \qquad \int_0^{\pi}\left(\int_0^{\pi/2} y\cos(x+y)\,dy\right)dx$$

$$\int_0^{\pi/2}\left[\ y\sin(x+y)\Big|_0^{\pi}\right]dy$$

$$\int_0^{\pi/2}\left[y\sin(\pi+y) - y\sin(y)\right]dy$$

$$\int_0^{\pi/2}\left[y\ (\sin(\pi+y) - \sin(y))\right]dy \qquad \text{By PARTS!}$$

$$u = y \qquad dv = \sin(\pi+y) - \sin(y)\,dy$$

$$du = dy \qquad v = -\cos(\pi+y) + \cos(y)$$

$$y\left(-\cos(\pi+y) + \cos(y)\right)\Big|_0^{\pi/2} - \int_0^{\pi/2} -\cos(\pi+y) + \cos(y)\,dy$$

$$\tfrac{\pi}{2}\left(-\cos\left(\tfrac{3\pi}{2}\right) + \cos\left(\tfrac{\pi}{2}\right)\right) - 0\left(-\cos(\pi) + \cos(0)\right] - \left[-\sin(\pi+y) + \sin(y)\Big|_0^{\pi/2}\right]$$

$$\tfrac{\pi}{2}(-0+0) - 0(1+1) \quad = - \left[\left(-\sin\left(3\pi/2\right) + \sin(\pi/2)\right) - \left(-\sin(\pi) + \sin(0)\right)\right]$$

$$= 0 \ ,\ - \left[(-(-1)+1) - 0\right] = \boxed{-2}$$

15.2 Double Integrals over General Regions

For the rectangular region, *R*, given by

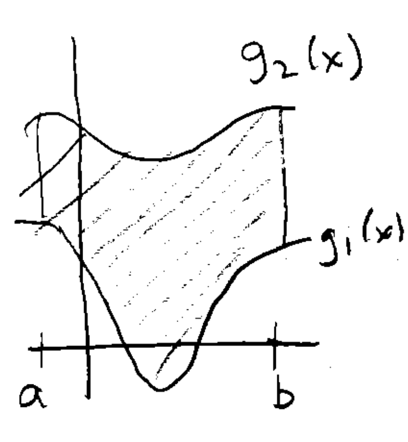
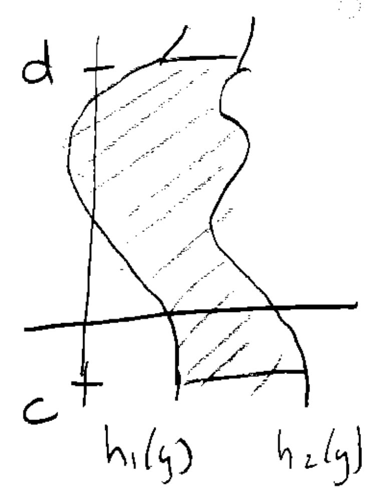
$$a \le x \le b \ , \qquad c \le y \le d$$

we learned:

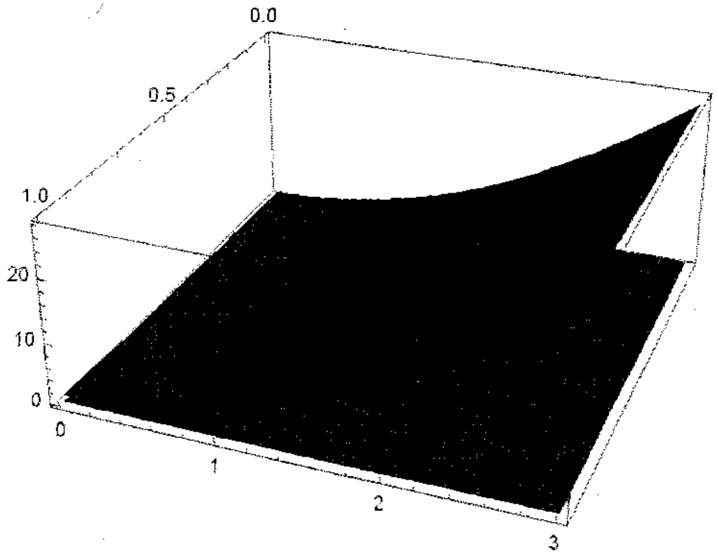
$$\iint\limits_R f(x,y)dA = \int_a^b \left(\int_c^d f(x,y)\,dy \right) dx$$

$$= \int_c^d \left(\int_a^b f(x,y)\,dx \right) dy$$

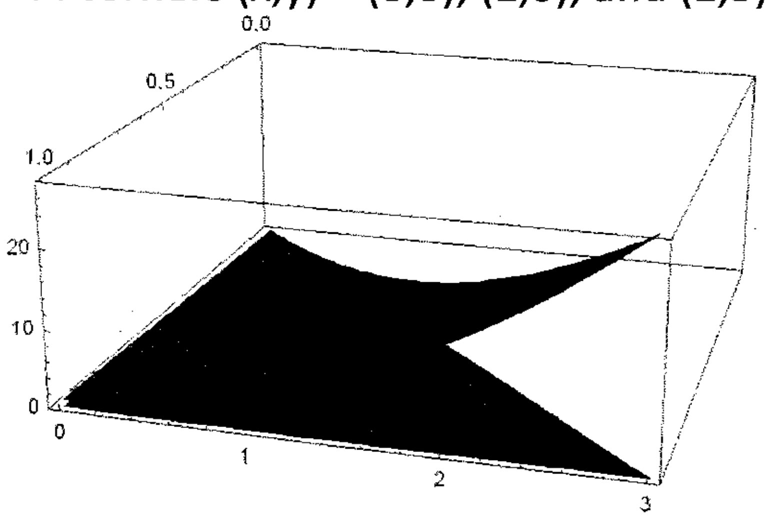
In 15.2, we discuss regions, *R*, other than rectangles.

<table>
<tr><td>Type 1 Regions
(Top/Bot)</td><td>Type 2 Regions
(Left/Right)</td></tr>
<tr><td>
</td><td>
</td></tr>
<tr><td>Given x in the range,
a ≤ x ≤ b, we have
$g_1(x) \le y \le g_2(x)$</td><td>Given y in the range,
c ≤ y ≤ d, we have
$h_1(y) \le x \le h_2(y)$</td></tr>
<tr><td>$\int_a^b \left(\int_{g_1(x)}^{g_2(x)} f(x,y)\,dy \right) dx$</td><td>$\int_c^d \left(\int_{h_1(y)}^{h_2(y)} f(x,y)\,dx \right) dy$</td></tr>
</table>

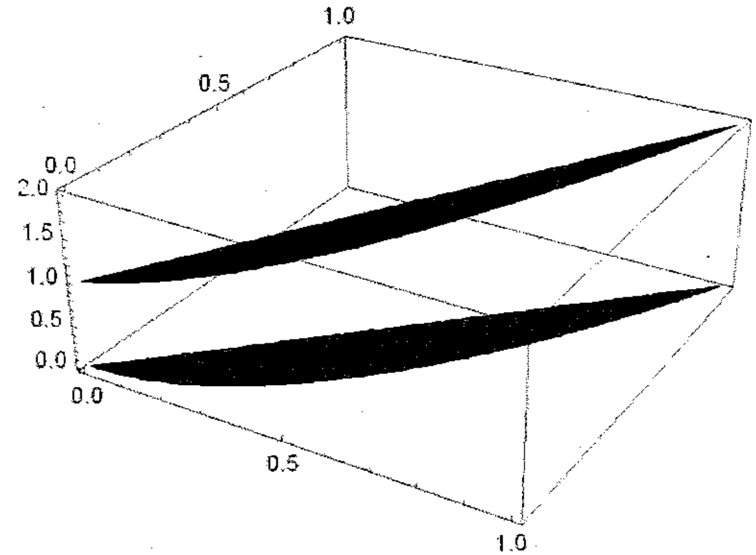
The surface $z = x + 3y^2$ over the rectangular region $R = [0,1] \times [0,3]$



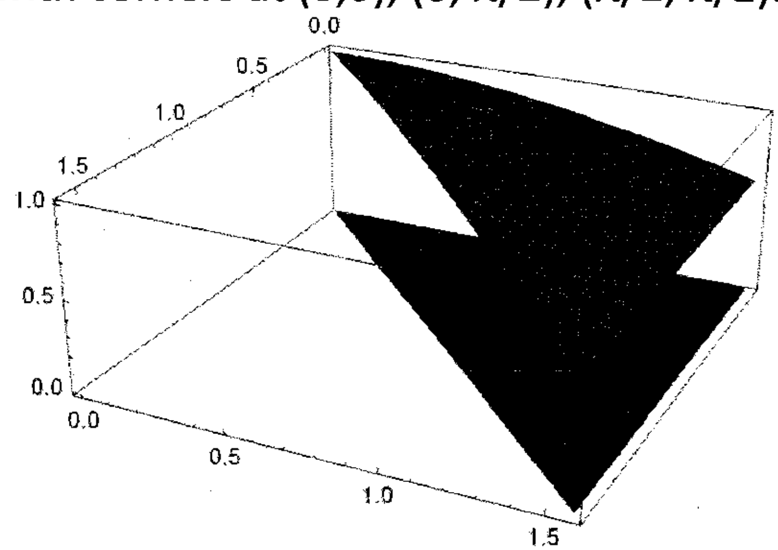
The surface $z = x + 3y^2$ over the triangular region with corners $(x,y) = (0,0)$, $(1,0)$, and $(1,3)$.



The surface $z = x + 1$ over the region bounded by $y = x$ and $y = x^2$.

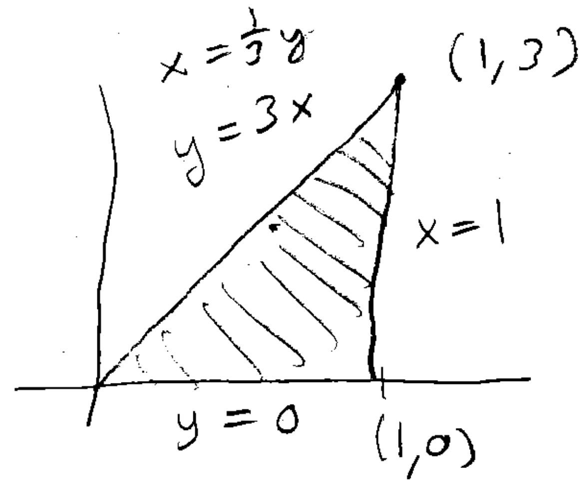


The surface $z = \sin(y)/y$ over the triangular region with corners at $(0,0)$, $(0, \pi/2)$, $(\pi/2, \pi/2)$.



Examples:

1. Let D be the triangular region in the xy-plane with corners (0,0), (1,0), (1,3).



$$\text{Evaluate} \iint_D x + 3y^2 dA$$

OPTION 1: FIX x
For some $0 \le x \le 1$

$\Rightarrow 0 \le y \le 3x$

$y = 0$ IS ALWAYS THE "BOTTOM" BOUND
$y = 3x$ IS ALWAYS THE "TOP" BOUND

OPTION 2: FIX y
for some $0 \le y \le 3$

$\Rightarrow \frac{1}{3}y \le x \le 1$

$x = \frac{1}{3}y$ IS ALWAYS THE "LEFT" BOUND
$x = 1$ IS ALWAYS THE "RIGHT" BOUND

$$\int_0^1 \left(\int_0^{3x} x + 3y^2 \, dy \right) dx \quad \text{OR} \quad \int_0^3 \left(\int_{\frac{1}{3}y}^{1} x + 3y^2 \, dx \right) dy$$